# NONLINEAR EFFECTS OF THE EKMAN LAYER ON THE DYNAMICS OF LARGE-SCALE EDDIES IN SHALLOW WATER

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Two-dimensional equations describing vortical flows in thin horizontal fluid layers are known as shallow-water equations [1]. These equations are valid for large-scale motions when the effect of nonuniformity of the flows in the vertical direction is unimportant. Attempts to more adequately describe horizontal layers within the framework of this approximation have led to a need to account for features of the structure of hydrodynamic fields across the layer. One method of deriving two-dimensional equations that describe essentially three-dimensional flows was developed in [2, 3]. In these studies, the hydrodynamic fields were represented by a finite Taylor series in horizontal coordinates, while the transverse profiles of the fields were determined from exact similarity solutions of the initial equations.

The main velocity discontinuities are concentrated in viscous boundary layers formed near solid horizontal boundaries. The effect of asymptotically thin viscous boundary layers on the evolution of vorticity usually reduces to additional energy dissipation, which can be described by means of linear friction. It was shown in [4-6] that such a parameterization satisfactorily describes actual experiments in a nonrotating fluid. This includes the case of the presence of a transverse magnetic field, when a Hartmann boundary layer develops near the boundaries [6]. The obviousness of this proposition has led investigators to generalize the linear friction model to rotating systems [1]. However, Ekman layers have significant differences from the boundary layers in nonrotating systems [7]. Specifically, the directions of motion of the fluid inside and outside the core of the flow do not coincide in the former case. It is shown below that this leads to transport of the vorticity of the average flow in the direction perpendicular to the direction of velocity averaged over the thickness of the layer. This effect makes the friction coefficient dependent on the rate of vortical motion and ensures the preferential propagation of cyclonic vortices ($\omega/f > 0$, where $\omega$ is vorticity and f is the doubled frequency of rotation of the layer). The character of evolution of a specific vortex also depends on the local structure of the large-scale velocity field present as the background when the vortex is generated. The dependence of the thickness of the Ekman layer on the local vorticity of the flow was noted in [8].

We will examine a thin horizontal layer of fluid rotating about a vertical axis with the angular velocity f/2. We write the equations of motion of the fluid as

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + f\mathbf{n}\times\mathbf{v} = -\frac{1}{\rho}\nabla P - g\mathbf{n} + \nu\left(\Delta\mathbf{v} + \partial_z^2\mathbf{v}\right),\ \nabla\cdot\mathbf{v} = 0, \qquad (1)$$

where $\mathbf{v}$ is the velocity with the components u, v, and w along the axes x, y, and z, respectively; $\mathbf{n}$ is the vector of a normal directed along the z axis; P is pressure; $\rho$ is density; g is acceleration due to gravity; $\nu$ is the viscosity coefficient. The following conditions are satisfied at the horizontal boundaries of the layer:

$$v = u = w = 0,\ z = 0,\ \partial_z v = \partial_z u = w = 0,\ z = h. \qquad (2)$$

Equations (1) are widely used in geophysics problems to describe motion in thin spherical layers of fluid on the surface of self-gravitating bodies when the dimensions of the region are considerably less than the radius of the sphere. In this case, the centrifugal forces are offset by the component of the gravitational field directed along the horizontal boundary, while the Coriolis parameter f is equal to the projection of angular velocity on the normal to the surface (the so-called f-plane approximation [1]). The adopted boundary conditions, excluding vertical velocity on the top free boundary, impose certain limitations on the characteristic times $\tau$ and scales $\ell$ of the processes in question. We actually ignore the motions connected with gravity waves on the surface of the fluid and motions whose scale is comparable to the thickness of the layer. Thus,

Perm'. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 49-54, March-April, 1991. Original article submitted May 31, 1989; revision submitted November 15, 1989.

$$\tau \gg 1/f,\ h \ll l \ll L_R = \sqrt{gh}/f. \qquad (3)$$

Due to the nondeformability of the top boundary, the horizontal velocity averaged across the layer is nondivergent: $u = -\partial_y\psi$, $v = \partial_x\psi$. After Eq. (1) is averaged over z and pressure is excluded, this fact makes it possible to obtain the following equation for vorticity

$$\partial_t\Delta\psi + \{\psi,\ \Delta\psi\} = \nu\Delta^2\psi + \Phi. \qquad (4)$$

Here, $\{\psi,\ \Delta\psi\} = \partial_x\psi\partial_y\Delta\psi - \partial_y\psi\partial_x\Delta\psi$ are Poisson brackets; $\Delta = \partial_x^2 + \partial_y^2$ is the two-dimensional Laplace operator; $\Phi$ is a functional;

$$\Phi = \int_0^h \left[\partial_x\left(\nu\partial_z^2 v - u\partial_x v - v\partial_y v\right) - \partial_y\left(\nu\partial_z^2 u - u\partial_x u - v\partial_y u\right)\right] dz + \{\psi,\ \Delta\psi\}.$$

To close (4), it is necessary to find the dependence of $\Phi$ on the stream function $\psi$. In accordance with [3], we introduce the small parameter $\delta = h/\ell$ (where $\ell$ is the characteristic horizontal scale of the flows being studied) and we expand the stream function into a Taylor series in the coordinates x and y.

For large-scale flows ($\delta \ll 1$), to within $O(\delta^2)$ we can limit ourselves to linear terms of the expansion. The structure of the flows u, v in this case is determined by the exact solution first obtained by Ekman. For boundary conditions (2), this solution has the form

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \mp\,\partial_{y,x}\psi + e^{-kz/h}\left[c_{1,2}\cos\frac{kz}{h} \pm c_{2,1}\sin\frac{kz}{h}\right] + e^{kz/h}\left[\pm c_{3,4}\cos\frac{kz}{h} + c_{4,3}\sin\frac{kz}{h}\right],$$

where $k^{-1} = h_E/h$ is the relative thickness of the boundary layer; $h_E = \sqrt{f/2\nu}$. The coefficients $c_k$ are determined as

$$2c_1 = -\varepsilon_1\partial_y\psi + (1 - \varepsilon_2)\partial_x\psi,\ 2c_2 = (1 - \varepsilon_2)\partial_y\psi - \varepsilon_1\partial_x\psi,$$
$$2c_3 = \varepsilon_1\partial_y\psi - (1 + \varepsilon_2)\partial_x\psi,\ 2c_4 = -(1 + \varepsilon_2)\partial_y\psi - \varepsilon_1\partial_x\psi,$$
$$\varepsilon\varepsilon_1 = \cos k \sin k,\ \varepsilon\varepsilon_2 = \mathrm{ch}\,k\,\mathrm{sh}\,k,\ \varepsilon = \cos^2 k\,\mathrm{ch}^2 k + \sin^2 k\,\mathrm{sh}^2 k.$$

The solution is valid when the role of the nonlinear terms is small. This situation corresponds to the requirement $Ro = U/f\ell \ll 1$ (where Ro is the Rossby number and U is the characteristic velocity). This estimate is equivalent to condition (3), which follows from the requirement that the top boundary be nondeformable. Use of the Ekman solution for approximation of the velocity profile makes it possible to calculate the functional $\Phi$ and close Eq. (4). For rapid rotation ($k \gg 1$, or else a pronounced Ekman layer will be absent) we have

$$\Phi = \frac{6}{k}\{\psi,\ \Delta\psi\} + \frac{2}{k}\nabla(\Delta\psi\nabla\psi) - \frac{f}{2k}\Delta\psi. \qquad (5)$$

The first term makes a small addition to the nonlinear term in the left side of (4), while the other two terms are of the same order of smallness at $\Delta\psi \sim f$. Inserting of (5) into (4) yields

$$\partial_t\Delta\psi + \gamma\{\psi,\ \Delta\psi\} = \nu\Delta^2\psi - \nabla\varkappa\nabla\psi. \qquad (6)$$

Here, $\varkappa$ is the nonlinear friction coefficient, connected with vorticity: $\varkappa = \mu(1 - \alpha\Delta\psi)$, where $\mu = f/2k$, $\gamma = 1 - 6/k$, $\alpha = 4/f$. At $\Delta\psi \ll f$, the coefficient $\varkappa \simeq \mu$ and coincides with the usual parameterization of linear friction.

Two effects follow from the form of Eqs. (6) and should be manifested in thin rotating layers of fluid. First, vortices of different signs are subject to viscous dissipation in the boundary layer to different degrees. This should result in the preferential propagation of cyclones ($\Delta\psi/f > 0$). Second, the quantity $\varkappa$ becomes negative for sufficiently intense cyclonic vortices, and this situation corresponds to the addition of energy to vortices of the given scale. It should be noted that both cyclone and anticyclone asymmetry and the intensification of large-scale vortices (known as the effect of negative viscosity [1]) take place in geophysical flows.

In order to study the properties of the solutions of model equation (6), it is convenient to change over to dimensionless variables. Here, we use $\sqrt{\nu/\mu}$, $1/\mu$, $\nu/\mu\alpha$ as the units of measurement of length, time, and the stream function. Then (6) takes the form

$$\partial_t\omega + \frac{2\gamma}{k}\{\psi,\ \omega\} = \nabla[\nabla\omega - (1 - \omega)\nabla\psi],\ \omega = \Delta\psi \qquad (7)$$

(k is the inverse relative thickness of the boundary layer). The simplest solution of (7) describes flows with uniform vorticity, the magnitude of which depends on the time as $\omega = \omega_0[\omega_0 + (1 - \omega_0)e^{t-t_0}]^{-1}$, where $\omega_0$ is vorticity at the initial moment of time $t = t_0$. Steady-state solutions exist only at $\omega_0 = 0$ and $\omega_0 = 1$. If $\omega_0 \in (1, -\infty)$, then vorticity decreases monotonically to zero, while at $\omega_0 > 1$ it becomes infinitely large in a finite period of time. The latter is evidence of the presence of a mechanism of instability and the possible existence of isolated vortex solutions.

As can easily be checked by direct substitution, steady-state solutions of Eq. (7) satisfy the nonlinear two-dimensional Klein–Gordon equation

$$\omega = 1 + Ce^{-\psi} \tag{8}$$

(where C is an arbitrary constant).

If we limit ourselves to the case of unidimensional jet flows $\psi = \psi(y)$, then (8) can be integrated exactly:

$$\pm \sqrt{2}\, y = \int \frac{d\psi}{\sqrt{\psi - Ce^{-\psi}}}. \tag{9}$$

Integral (9) is not expressed in elementary functions, but it nonetheless makes it possible to fully explain the properties of the solutions obtained here. In the given case of unidimensional jets, Eq. (8) describes cyclonic solitons existing against a background of flow with constant shear ($\omega = 1$). For weak and strong perturbations, the structure of solitons is given by the following asymptotic expressions:

$$\omega = 1 + \varepsilon e^{-y^2/2}, \quad \omega = 1 + \frac{2}{\varepsilon^2 \operatorname{ch}^2 (y/\varepsilon)}, \quad |\varepsilon| \ll 1.$$

Anticyclonic solutions are possible only in the presence of velocity discontinuities in the flow ($\omega = 1 - 2k^2 \cos^{-2} ky$). The characteristic scale of weak solitons is equal to unity, while for strong perturbations of the steady background flow ($\omega = 1$) they are inversely proportional to the root of the amplitude of the perturbation. In dimensional variables, the characteristic scale of a solitary jet is equal to $\sqrt{\nu/\mu}$, which corresponds to the scales of viscous boundary layers in ordinary shallow water [1]. It should be noted that the parameter $\sqrt{\nu/\mu h^2}$, characterizing the relative dimensions of solitons, is much less than unity for laminar flows and much greater than unity when turbulent transport coefficients $\nu = \nu_t$ was used [1]. This suggests that the proposed theory might be valid for turbulent flows.

Let us now examine the effect of nonlinear vortical flow on the behavior of medium-scale turbulence in a rotating fluid layer. In the case when the layer is also turbulent through its thickness, the phenomenon of the transport of vorticity across streamlines remains in effect. The difference is that the quantiy $k = h/h_E$ is determined as the actually observed relative thickness of the turbulent Ekman layer. Turbulent flows of this type have relevance to geostrophic flows.

Large-scale geostrophic turbulence is quasi-two-dimensional, but the presence of such specific factors as the β-effect, baroclinic distribution, and stratification make it difficult to interpret its properties from the standpoint of two-dimensional turbulence [9]. The factor in question is in a certain sense the simplest factor – it is connected only with dissipative effects at the boundary and is manifest in any rotating layer.

The effect of the linear dissipative term on the properties of two-dimensional turbulent flows has been studied previously by numerically modeling two-dimensional turbulence [10]. Lilly [10] demonstrated the possibility of obtaining an interval of entropy transfer to small scales with the spectral energy distribution $E(k) \sim k^{-3}$ (where k is the wave number) in a system with linear friction. Sommeria [6] experimentally studied the effect of linear friction on the inertial interval of energy transfer $E(k) \sim k^{-5/3}$ with the inverse energy cascade characteristic of two-dimensional turbulence. Conducting tests involving the placement of a layer of mercury in a transverse magnetic field, Summeria [6] not only established the presence of an inverse cascade, but also obtained the dependence of the Kolmogorov constant on the magnitude of the linear friction determined by the magnetic field.

Thorough study of the properties of turbulent regimes associated with Eq. (6) requires the realization of numerical experiments on supercomputers and is not part of the goal of the present investigation. At the same time, some of the common properties of uniform turbulent flows that can be described by such equations can be examined on the basis of simple small-parameter models.

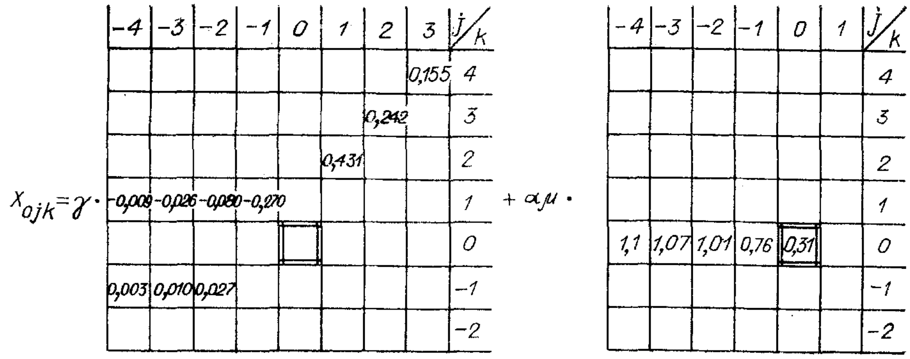


Fig. 1

The main features of processes involving redistribution of energy between motions taking place at different scales can be described by means of cascade equations for collective variables $A_i$, each of which characterizes the pulsations of the velocity field in a certain interval of wave numbers. Cascade equations minimize the number of dimensions of systems that describe turbulent flows within a wide range of wave numbers. These equations have the form

$$\partial_t A_i = \sum_{jk} X_{ijk} A_j A_k + Y_i A_i + f_i, \tag{10}$$

where $f_i$ characterizes the energy sources in the respective intervals of the spectrum. Cascade models of the type in [10] have been constructed by a number of investigators (see [11-13], for example). While differing in the methods of subdivision of the wave-vector space and determination of the variables $A_i$, the restrictions imposed on triad interactions, and other details, these models retain the basic properties of the initial equations of motion of the fluid: they satisfy the conservation laws in the nondissipative limit and adequately describe nonlinear interactions between modes.

One method of obtaining Eqs. (10) consists of the following [13, 14]. The space of wave numbers is divided into octaves $k_i = 2k_{i-1}$, and two-dimensional turbulent eddies are described by means of basis functions $\psi_i(\mathbf{r} - \mathbf{r}_i)$ such that their spectrum is localized in the corresponding ring $k_{i-1} < |\mathbf{k}| < k_i$. Here, $\mathbf{r}_i$ is the position vector of the center of the vortex, with an increase in the subscript i by unity corresponding to halving of the size of the vortex. We have the following for a fixed triad of vortices and initial equations (6)

$$y_i = \int \Delta\psi_i [\nu\Delta^2\psi_i - \mu\Delta\psi_i]\, d\mathbf{r},$$

$$x_{ijk} = -\int \Delta\psi_i [\gamma\{\psi_j, \Delta\psi_k\} + \alpha\mu(\Delta\psi_j\Delta\psi_k + \nabla\psi_j\nabla\Delta\psi_k)]\, d\mathbf{r}.$$

The quantity $A_i$ is a collective characteristic of all vortices of the i-th dimension. Thus, the energy density for vortices of this scale $E_i = \langle A_i^2 \rangle$. The elements of the matrix $X_{ijk}$ are determined from the mean-square values of $x_{ijk}$ obtained for all possible mutual locations of the vortices i, j, k and from the conditions of energy and entropy conservation in the inviscid limit [15].

The characteristic differences between the turbulent regimes of Eqs. (6) and ordinary shallow water can be discerned from the form of the matrix $X_{ijk}$ in the respective cases. The structure of the matrix is illustrated in Fig. 1, which shows the central part of the matrix for i = 0 (the remaining elements can be obtained from the relation $X_{ijk} = 2^i X_{0,j-i,k-i}$) [13]. The left side of the matrix describes the behavior of two-dimensional turbulence [15] and at $Y_i = 0$ satisfies energy and entropy conservation conditions (10). It should be noted that the matrix contains no diagonal terms, i.e., the elements of the matrix are equal to zero when any pair of induces coincides.

With $Y_i \neq 0$ but a value of zero for α, Eqs. (10) describe two-dimensional turbulence with linear friction. The spectral properties for this case are shown in Fig. 2. If motion is excited at scales characterized by the wave number k*, a spectrum of the form $E(k) \sim k^{-3}$ is established to the right and is accompanied by transport of vorticity to small-scale motion. At the same time, an inverse energy cascade with the distribution $E(k) \sim k^{5/3}$ is realized for k < k*. The energy reaches a maximum value at k = k**. The position of the maximum is determined by the power of the energy sources and the efficiency of the linear friction. In the low-frequency part (k < k**), the spectrum approaches the relation $E(k) \sim k$.

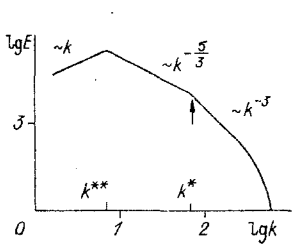


Fig. 2

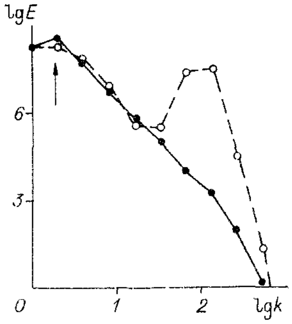


Fig. 3

The second part of the matrix $X_{ijk}$ becomes important at $\alpha \neq 0$ (see Fig. 1). The most important elements of this part are the diagonals $X_{iji}$, which rapidly move toward the asymptote $X_{0j0} = 1.1\alpha$ with an increase in the difference $i - j$. With allowance for the realistically attainable relations between the coefficients $\alpha$ and $\gamma$, the remaining elements of the matrix are unimportant and are omitted from Fig. 1.

The numerical solution of Eqs. (10) for different values of $\alpha$ showed that the integral properties of turbulent flows depend weakly on $\alpha$. With an arbitrary set of signs for the variables $A_i$, the equations still allow a solution corresponding to the law $E(k) \sim k^{-3}$ and an inverse energy cascade is still realized at $k < k^*$. A maximum remains in the time-averaged spectra, the position of this maximum being nearly independent of $\alpha$. At the same time, the local properties of the flow under significant changes.

In regimes corresponding to developed turbulence, Eqs. (10) have no stable steady-state solutions. The values of $A_i$ undergo random oscillations, while the spectral distributions are obtained by calculating the time-averaged values of $\langle A_i^2 \rangle$ [13, 14]. A specific vortex evolves in the flow against the background of total vorticity created by all of the large vortices. If we allow for this effect, we obtain a nondecreasing diagonal $X_{iji}$ in (10). This in turn leads to a situation whereby the character of development of the given quantity $A_i$ depends to a significant extent on how the directions of rotation are composed, i.e., on the signs throughout the chain of $A_j$ for $j \leqslant i$. At moments when the chain is made up of positive $A_j$, the vortex may receive an appreciable portion of energy. When negative values predominate, there is an increase in energy dissipation.

Figure 3 shows sample calculations corresponding to the excitation of large-scale motion. At $\alpha = 0$, the regime established in the system yields a spectrum which is close to the law $k^{-3}$ (solid line) within a considerably broad range of wave numbers $k > k^*$. The dashed line shows one of the motion energy surges obtained in the intermediate scales for the same conditions of pumping of energy into the motion and $\alpha\mu = 0.1$. The time peaks in the solution of the cascade equations correspond to pronounced spatial alternation of the flows described by Eq. (6).

The results obtained here are consistent with generally accepted reprsentations on the character of geostrophic turbulence [1, 9]. An inverse cascade of energy to the macroscales takes place in the flow, but this phenomenon does not determine the dynamics of the energy-containing structures of the mesoscales. The fact of the mesoscales is determined in large part by the specific structure of the large-scale field of vorticity. Here, the vortices which develop are primarily cyclonic in character.

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# SLOW MOTIONS OF A SOLID IN A CONTINUOUSLY STRATIFIED FLUID

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UDC 532.5

The problem of the motion of a solid in a fluid of nonuniform density (stratified) is a particularly difficult and intensively investigated one [1]. The almost total absence of exact solutions has led to a great deal of work on the development of linear-approximation models [2-5].

Again in the linear approximation we have constructed particular solutions of the flow problem for the slow motions of a three-dimensional or two-dimensional solid in an ideal incompressible stratified fluid. The shape of the body and its direction of motion may be arbitrary, and the dependence of the velocity on time t has a special form [proportional to $\exp(\alpha t)$ with constant $\alpha > 0$]. A new method of constructing the solution is proposed. It is based on the following remarkable fact: by direct transformation the problem can be reduced to the classical problem of the potential flow of a homogeneous fluid past some other fictitious body. This equivalence makes it possible to calculate the velocity and resistance fields in the stratified fluid. And the formulas for the resistance are simple analytic expressions.

The limiting solutions as $\alpha \to 0$, which are of interest from two points of view, have been studied in detail. First, they correspond to the important practical case of uniform motion, and, second, they coincide with the solutions of the problem of the instantaneous setting in motion of a body initially at rest. At the same time, the problem of impulsive motion, previously considered in various particular formulations [3, 5], has been solved in general form. The calculations showed that the limiting ($\alpha \to 0$) flows have a characteristic layered structure. The vertical velocity component is equal to zero, and the fluid moves in horizontal layers ($z$ = const). In all cases the resistance to the uniform motion of a three-dimensional body (less the buoyancy force) is equal to zero, which gives a result analogous to the D'Alembert paradox. For a two-dimensional body a fundamentally different answer is obtained: in the limit as $\alpha \to 0$ the resistance is finite for both horizontal and vertical motion.

Thus, a number of general results relating to low Froude number regimes have been obtained for stratified flow past a body. The analogous problem of the motion of a body in a rotating fluid was solved in [6]. In the light of the analogy between stratification and rotation [7, 8] our results are a development of the approach adopted in [6].

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Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 55-60, March-April, 1991. Original article submitted July 3, 1989; revision submitted November 15, 1989.

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